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Some quantum Lie algebras of type D_n positive

César Bautista[1] **and María Araceli Juaréz-Ramírez**[2]

[1] Facultad de Ciencias de la Computación, Benemérita Universidad Autónoma de Puebla, Edif 135, 14 sur y Av San Claudio, Ciudad Universitaria, Puebla Pue. CP 72570, Mexico
[2] Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Edif 158 Av San Claudio y Rio Verde sn Ciudad Universitaria, Puebla Pue. CP 72570, Mexico

E-mail: bautista@cs.buap.mx and Araceli.Juarez@fcfcm.buap.mx

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Abstract

A quantum Lie algebra is constructed within the positive part of the Drinfeld–Jimbo quantum group of type D_n. Our quantum Lie algebra structure includes a generalized antisymmetry property and a generalized Jacobi identity closely related to the braid equation. A generalized universal enveloping algebra of our quantum Lie algebra of type D_n positive is proved to be the Drinfeld–Jimbo quantum group of the same type. The existence of such a generalized Lie algebra is reduced to an integer programming problem. Moreover, when the integer programming problem is feasible we show, by means of the generalized Jacobi identity, that the Poincaré–Birkhoff–Witt theorem (basis) is still true.

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1. Introduction

In [1] Lyubashenko and Sudbery called for generalizations of Lie algebras. The main problem is to find a finite-dimensional vector space within the Drinfeld–Jimbo quantum groups [2, 3] with a binary operation satisfying generalized antisymmetry and generalized Jacobi identity in such a way that its universal enveloping algebra coincides with the quantum group; such structures will be called *quantum Lie algebras*. Some additional constraints are required, for instance, the existence of a Poincaré–Birkhoff–Witt basis or the adjoint map constructed by commutators and by the Hopf algebra structure.

Delius *et al* [4, 5] suggested using quantum Lie algebras for the understanding of some properties (quantum mass or charge ratios and exact S-matrices for elementary particles) of quantum integrable models, since integrable models can be studied after classical Lie algebras. Similarly, a q-deformed gauge theory is proposed by Sudbery [6], based on quantum Lie algebras of type A_n; extension to other classes of quantum Lie algebras (like the one proposed here) remains to be investigated.

Conformal field theories in two dimensions, which give the low energy limit of string theory, have been found to have quantum group symmetries related to monodromy transformations in Wess–Zumino–Witten models, for $SU(2)$ in [7] and then generalized for semi-simple algebras in [8]. In these works some kind of generalized commutators (q-commutators) appear. A natural goal is to find the properties of such q-commutators under the guide of classical Lie algebras.

Another application of the quantum Lie algebra formalism appears in the theory of error correction codes in quantum computation [9].

From a mathematical point of view, the quantum Lie algebras are useful because they allow us to reduce problems on the infinite-dimensional quantum group to the finite-dimensional quantum Lie algebra [10]. For instance, we show that the problem of finding Poincaré–Birkhoff–Witt bases in the positive part of the Drinfeld–Jimbo quantum group of type D_n positive is reduced to finding a representation of our finite-dimensional quantum Lie algebra on the related q-symmetric algebra.

Just like the quantum groups, there are two classes of quantum Lie algebras; those related to the Woronowicz quantum groups and those related to Drinfeld–Jimbo quantum groups. While the first have well-established generalizations of the antisymmetry property and the Jacobi identity, in the latter case different notions of quantum antisymmetry and quantum Jacobi identity appear. One might say this is why the theory of Woronowicz's quantum Lie algebras has got more attention than that of Drinfeld–Jimbo. Because, after all, quantum antisymmetry and quantum Jacobi identity could lead to a quantum Lie algebra theory by analogy (or deformation) to the classical case.

In this paper we are interested in the construction of a quantum Lie algebra by means of q-commutators for $U_q^+\mathfrak{o}(2n)$ the positive part of the Drinfeld–Jimbo quantum group of type D_n. We show that such a construction is feasible and we obtain a finite-dimensional vector space with an additional structure satisfying generalized antisymmetry and a generalized Jacobi identity such that its universal enveloping algebra (generalized) is $U_q^+\mathfrak{o}(2n)$. But, in addition to a binary operation, which is a quadratic–linear operator, we obtain two other operators, which are quadratic–quadratic and quadratic–cubic, respectively. However, we can prove the Poincaré–Birkhoff–Witt theorem over such structures. Further, since a bilinear form is a quadratic–scalar operator, our results can be applied to Clifford algebras.

In order to keep the notation under control, we make use of some diagrammatic notation similar to the quivers (oriented graphs) appearing in the theory of representation of finite algebras [11].

Quivers related to quantum groups were used by Cibils [12, 13] and Ringel [14]. However, in this paper our approach is different. We start from a generalized Jacobi identity for the generalized commutator $f(x, y) = xy - \sigma(xy)$ which is valid in every associative algebra analogous to the Jacobi identity that becomes the associative algebra, a Lie algebra with commutator $[x, y] = xy - yx$.

The properties of our quantum $\mathfrak{o}^+(2n)$ are proved by mathematical induction over n. First, the properties for ($n = 4$) quantum $\mathfrak{o}^+(8)$ follow by straightforward calculations, then quantum $\mathfrak{o}^+(2n)$ for $n > 4$ satisfies the same equations as quantum $\mathfrak{o}^+(8)$ because, in some sense, quantum $\mathfrak{o}(2n)$ is covered by copies of quantum $\mathfrak{o}^+(8)$ (see theorem 2).

In the literature there are already some proposals for quantum Lie algebras of type D_n: quantum Lie algebras defined by a generalization of the Friedrich criterion (characterization of the Lie algebra structure by primitives) from Kharchenko [15], but without antisymmetry or a Jacobi identity; a quantum Lie algebra due to Delius *et al* [16] with an antisymmetry but without a Jacobi identity, among others.

The organization of this paper is as follows. We start by applying a generalized Jacobi identity to some generators of $U_q^+(sl_4)$, the positive part of the Drinfeld–Jimbo quantum group of type A_3. In section 3 we define the structure of quantum $\mathfrak{o}(2n)$ and prove that, in some sense, $U_q^+\mathfrak{o}(8)$ is a universal enveloping algebra of quantum $\mathfrak{o}^+(8)$. A generalization for this result is proved in section 4. The definition of a generalized Lie algebra (called σ-Lie algebra), which also generalizes classical Clifford algebras, as well as a representation of our generalized Lie algebra on the q-symmetric algebra, just like in the classical case, is given in section 5. In section 6, we define morphisms of σ-Lie algebras and explain why the proof that quantum $\mathfrak{o}(2n)$ is a σ-Lie algebra, is an integer programming problem. Finally, in section 7, we discuss the differences between quantum Lie algebras from [10, 16] and ours.

2. Non-braided identities and quantum $U^+(sl_4)$

Our starting point is the identity (1), which holds in every associative algebra. The notation is the following. Assume that F is a commutative ring, L, M, N three F-modules and $f : M \to N$ an F-linear map. We denote $f_1 = f \otimes Id : M \otimes L \to N \otimes L$ and $f_2 = Id \otimes f : L \otimes M \to L \otimes N$ where $Id : L \to L$ is the identity map.

Proposition 1. *Let A be an associative algebra, M a submodule of the module A, m the multiplication map of A and $\sigma : M \otimes M \to M \otimes M$ a linear map. Then, if $f = m(Id - \sigma) : M \otimes M \to A$, it follows*

$$f(f_1 - f_2) = m(f_1\sigma_2 - f_2\sigma_1 + \sigma f_2 - \sigma f_1 + f_2\sigma_1\sigma_2 - f_1\sigma_2\sigma_1) - mm_1(\sigma_2\sigma_1\sigma_2 - \sigma_1\sigma_2\sigma_1). \tag{1}$$

Proof. This proposition is proved by straightforward calculations. □

In fact, (1) is a generalization of the Jacobi identity, because it is equivalent to

$$\begin{aligned} f(f_2 - f_1 + f_1\sigma_2) = m\,&((f_1\sigma_2\sigma_1 + f_2\sigma_1 - \sigma f_2 - \sigma f_1\sigma_2) + (\sigma f_1 - f_2\sigma_1\sigma_2)) \\ &+ mm_1(\sigma_2\sigma_1\sigma_2 - \sigma_1\sigma_2\sigma_1) \end{aligned} \tag{2}$$

therefore, if σ stands for the usual flip $x \otimes y \mapsto y \otimes x$ and $M = A$ then (2) becomes the classical Jacobi identity. A more general case is when $M = A$ and σ is only required to satisfy the braid equation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$ and

$$\sigma f_2 + \sigma f_1\sigma_2 = f_2\sigma_1 + f_1\sigma_2\sigma_1 \qquad \text{and} \qquad \sigma f_1 = f_2\sigma_1\sigma_2 \tag{3}$$

then (2) becomes the generalized Jacobi identity discovered by Woronowicz within the frame of differential calculus on quantum groups [17]. Also, the conditions (3) appeared in [17]. The identity (1) was taken from [18].

It is well known that semi-simple Lie algebras are made of copies of sl_n pasted together with additional relations. In the quantum case, the Drinfeld–Jimbo quantum groups are made of copies of quantizations of the universal enveloping algebra of sl_n pasted together with additional relations. The following q-commutators in $U_q^+(sl_4)$ are a part of the relations of the Drinfeld–Jimbo quantum group of type D_n.

Lemma 1. *Let F be a commutative ring with unit and $q \in F^*$. Let $U_q^+(sl_4)$ be the associative F-algebra generated by α_i, $i = 1, 2, 3$, with relations*

$$\alpha_1\alpha_3 - \alpha_3\alpha_1 = 0 \qquad \alpha_i\alpha_j^2 - (q + q^{-1})\alpha_j\alpha_i\alpha_j + \alpha_j^2\alpha_i = 0 \qquad |i - j| = 1$$

then there exist generators $\alpha_{ij} \in U_q^+(sl_4)$, $1 \leqslant i < j \leqslant 4$, *with relations*

$$\begin{array}{ll}
m(\alpha_{12} \otimes \alpha_{13} - \sigma(\alpha_{12} \otimes \alpha_{13})) = 0 & m(\alpha_{12} \otimes \alpha_{23} - \sigma(\alpha_{12} \otimes \alpha_{23})) = \alpha_{13} \\
m(\alpha_{12} \otimes \alpha_{14} - \sigma(\alpha_{12} \otimes \alpha_{14})) = 0 & m(\alpha_{12} \otimes \alpha_{24} - \sigma(\alpha_{12} \otimes \alpha_{24})) = \alpha_{14} \\
m(\alpha_{12} \otimes \alpha_{34} - \sigma(\alpha_{12} \otimes \alpha_{34})) = 0 & m(\alpha_{13} \otimes \alpha_{23} - \sigma(\alpha_{13} \otimes \alpha_{23})) = 0 \\
m(\alpha_{13} \otimes \alpha_{14} - \sigma(\alpha_{13} \otimes \alpha_{14})) = 0 & m(\alpha_{13} \otimes \alpha_{34} - \sigma(\alpha_{13} \otimes \alpha_{34})) = \alpha_{14} \\
m(\alpha_{13} \otimes \alpha_{24} - \sigma(\alpha_{13} \otimes \alpha_{24})) = (q - q^{-1}) & m(\alpha_{23} \otimes \alpha_{14}) \\
m(\alpha_{23} \otimes \alpha_{14} - \sigma(\alpha_{23} \otimes \alpha_{14})) = 0 & m(\alpha_{23} \otimes \alpha_{24} - \sigma(\alpha_{23} \otimes \alpha_{24})) = 0 \\
m(\alpha_{23} \otimes \alpha_{34} - \sigma(\alpha_{23} \otimes \alpha_{34})) = \alpha_{24} & m(\alpha_{14} \otimes \alpha_{24} - \sigma(\alpha_{14} \otimes \alpha_{24})) = 0 \\
m(\alpha_{14} \otimes \alpha_{34} - \sigma(\alpha_{14} \otimes \alpha_{34})) = 0 & m(\alpha_{24} \otimes \alpha_{24} - \sigma(\alpha_{24} \otimes \alpha_{34})) = 0
\end{array}$$

where m *is the multiplication map of* $U_q^+(sl_4)$, $\sigma(\alpha_{ij} \otimes \alpha_{ab}) = q^{c_{ij,ab}}\alpha_{ab} \otimes \alpha_{ij}$, $c_{ij,ab} = \delta_{ia} - \delta_{ib} - \delta_{ja} + \delta_{jb}$ *and* δ *is the Kronecker delta.*

Proof. First we define $\alpha_{i(i+1)} = \alpha_i$, $i = 1, 2, 3$, and $\alpha_{13} = \alpha_{12}\alpha_{23} - q^{-1}\alpha_{23}\alpha_{12}$, $\alpha_{24} = \alpha_{23}\alpha_{34} - q^{-1}\alpha_{34}\alpha_{23}$. Then using (1) on $\alpha_{12} \otimes \alpha_{23} \otimes \alpha_{34}$, we get

$$\alpha_{13}\alpha_{34} - q^{-1}\alpha_{34}\alpha_{13} = \alpha_{12}\alpha_{24} - q^{-1}\alpha_{24}\alpha_{14}.$$

We put $\alpha_{14} = \alpha_{13}\alpha_{34} - q^{-1}\alpha_{34}\alpha_{13}$. Using again (1) on $\alpha_{12} \otimes \alpha_{23} \otimes \alpha_{24}$ and afterwards on $\alpha_{13} \otimes \alpha_{23} \otimes \alpha_{34}$, we obtain

$$\alpha_{13}\alpha_{24} - \alpha_{24}\alpha_{13} = (q - q^{-1})\alpha_{23}\alpha_{14}$$

and so on. □

Using the same technique one can prove

Lemma 2. *There exist generators* $\beta_{ij} \in U_q^+(sl_4)$, $1 \leqslant i < j \leqslant 4$, *with relations*

$$\begin{array}{ll}
\beta_{12}\beta_{13} - q^{-1}\beta_{13}\beta_{12} = 0 & \beta_{12}\beta_{23} - q\beta_{23}\beta_{12} = \beta_{13} \\
\beta_{12}\beta_{14} - q^{-1}\beta_{14}\beta_{12} = 0 & \beta_{12}\beta_{24} - q\beta_{24}\beta_{12} = \beta_{14} \\
\beta_{12}\beta_{34} - \beta_{34}\beta_{12} = 0 & \beta_{13}\beta_{23} - q^{-1}\beta_{23}\beta_{13} = 0 \\
\beta_{13}\beta_{14} - q\beta_{13}\beta_{23} = 0 & \beta_{13}\beta_{24} - \beta_{13}\beta_{24} = 0 \\
\beta_{13}\beta_{34} - q^{-1}\beta_{13}\beta_{34} = \beta_{14} & \beta_{23}\beta_{14} - \beta_{23}\beta_{14} = (q - q^{-1})\beta_{13}\beta_{24} \\
\beta_{23}\beta_{24} - q\beta_{23}\beta_{24} = 0 & \beta_{23}\beta_{34} - q^{-1}\beta_{23}\beta_{34} = \beta_{24}.
\end{array}$$

3. Quantum D_n positive

We are dealing with the Drinfeld–Jimbo quantum groups in Lusztig form [19] over commutative rings. This means that, if F is a commutative ring with unit, $q \in F^*$ and G is the Dynkin diagram of type D_n, $n \geqslant 4$, with nodes labelled $1, 2, \ldots, n$ and ramification node labelled $n - 2$, then the Drinfeld–Jimbo quantum group of type D_n positive, denoted as $U_q^+\mathfrak{o}(2n)$, is the associative F-algebra with 1, with generators $E_1, \ldots, E_n$ and with relations

$$\begin{array}{ll}
E_iE_j - E_jE_i = 0 & \text{if } i \text{ is not linked to } j \text{ in } G \\
E_iE_j^2 - (q + q^{-1})E_jE_iE_j + E_j^2E_i = 0 & \text{if } i \text{ is linked to } j \text{ in } G.
\end{array}$$

Let us take some formal letters, M_{ij} and S_{ij}, $1 \leqslant i < j \leqslant n$ (called canonical basic elements of $\mathfrak{o}^+(2n)_q$), ordered according to the following rules:

$$M_{i(i+1)} > \cdots > M_{in} > S_{in} > S_{i(i+1)} > \cdots > S_{i(n-1)} \qquad 1 \leqslant i \leqslant n-1 \tag{4}$$

$$S_{j(n-1)} > M_{(j+1)(j+2)} \qquad 1 \leqslant j \leqslant n-2. \tag{5}$$

Let L_n be the F-module with free basis given by the canonical basis of $\mathfrak{o}^+(2n)_q$. Now, we define ($1 \leqslant i < j \leqslant n$, $1 \leqslant a < b \leqslant n$)

$$\begin{aligned} d_{ij,ab} &= \delta_{ia} + \delta_{bj} + \delta_{ja} + \delta_{bi} \qquad & h_{ij,ab} &= \delta_{ai} - \delta_{aj} - \delta_{bj} + \delta_{bi} \\ c_{ij,ab} &= \delta_{ia} - \delta_{ib} - \delta_{ja} + \delta_{jb} \qquad & g_{ij,ab} &= \delta_{ai} - \delta_{bi} + \delta_{ja} - \delta_{jb} \end{aligned}$$

where δ stands for the Kronecker delta.

The non-associative algebra $\mathfrak{o}^+(2n)_q$ is a 5-tuple $(L_n, \sigma, B^{(1)}, B^{(2)}, B^{(3)})$ where σ and $B^{(1)}, B^{(2)}, B^{(3)}$ are linear maps; $\sigma : L_n \otimes L_n \to L_n \otimes L_n$, $B^{(k)} : L_n \otimes L_n \to L_n^{\otimes k}$, $k = 1, 2, 3$, defined by the conditions $\sigma^2 = Id$, $B^{(k)}\sigma = -B^{(k)}$ and

$$\begin{aligned} &\sigma(M_{ij} \otimes M_{ab}) = q^{c_{ij,ab}} M_{ab} \otimes M_{ij} \qquad && \text{if} \quad M_{ij} > M_{ab} \\ &\sigma(M_{ij} \otimes S_{ab}) = q^{h_{ij,ab}} S_{ab} \otimes M_{ij} && \text{if} \quad M_{ij} > S_{ab} \\ &\sigma(S_{ij} \otimes S_{ab}) = q^{d_{ij,ab}} S_{ab} \otimes S_{ij} && \text{if} \quad S_{ij} > S_{ab} \end{aligned}$$

as well as

$$B^{(1)}(M_{ij} \otimes M_{ab}) = \delta_{ja} M_{ib} - q\delta_{bi} M_{aj} \qquad B^{(1)}(M_{ij} \otimes S_{ab}) = \delta_{ja} S'_{ib} + \delta_{bj} S'_{ai}$$

$$B^{(1)}(S_{ij} \otimes S_{ab}) = 0$$

where

$$S'_{ij} = \begin{cases} S_{ij} & \text{if} \quad i < j \\ -qS_{ji} & \text{if} \quad i > j \\ 0 & \text{if} \quad i = j \end{cases}$$

and

$$B^{(2)}(M_{ij} \otimes S_{ab}) = \begin{cases} (q - q^{-1})q^{-1}(S_{ij} \otimes M_{ab} & \\ \quad -qS_{in} \otimes M_{an}) & \text{if} \quad i < a \quad j = b \\ -q(q - q^{-1})S_{in} \otimes M_{in} & \text{if} \quad i = a \quad j = b < n \\ (q - q^{-1})S_{in} \otimes M_{aj} & \text{if} \quad i < a < j < b = n \\ (q - q^{-1})S_{ib} \otimes M_{aj} & \text{if} \quad i < a < b < n \quad j = n \\ -(q - q^{-1})S_{ai} \otimes M_{bn} & \text{if} \quad a < b < i < j = n \\ 0 & \text{otherwise} \end{cases}$$

$$B^{(2)}(S_{ij} \otimes S_{ab}) = \begin{cases} -(q - q^{-1})S_{bn} \otimes S_{ai} & \text{if} \quad a < b < i < j = n \\ (q - q^{-1})S_{an} \otimes S_{ib} & \text{if} \quad i < a < b < j = n \\ 0 & \text{otherwise} \end{cases}$$

$$B^{(2)}(M_{ij} \otimes M_{ab}) = \begin{cases} (q - q^{-1})M_{aj} \otimes M_{ib} & \text{if} \quad i < a < j < b \\ 0 & \text{otherwise} \end{cases}$$

$$B^{(3)}(M_{ij} \otimes S_{ab}) = \begin{cases} -q(q - q^{-1})S_{in} \otimes M_{bj} \otimes M_{in} & \text{if} \quad i = a < b < j \\ 0 & \text{otherwise} \end{cases}$$

$$B^{(3)}(S_{ij} \otimes S_{ab}) = \begin{cases} -q(q - q^{-1})S_{ib} \otimes M_{jn} \otimes S_{jn} & \text{if} \quad i < j = a < b < n \\ 0 & \text{otherwise.} \end{cases}$$

Our main statement is that $\mathfrak{o}^+(2n)_q$ is a generalized Lie algebra with generalized universal enveloping isomorphic to the positive part of the Drinfeld–Jimbo quantum group of type D_n.

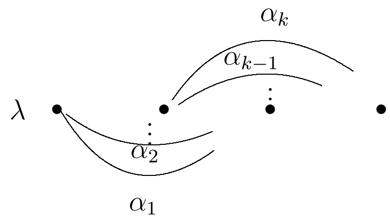
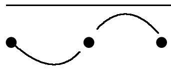
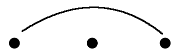
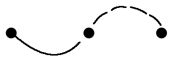
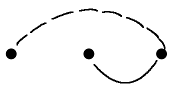
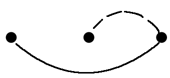


Figure 1. The tensorial product $\lambda\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k$.

Table 1. The map $B^{(1)}$. In this table and in the other tables the elements x, y are basic elements such that $x < y$.

$x \otimes y$	$B^{(1)}(x \otimes y)$
	
	
	
	$-q$ 
Otherwise	0

In order to prove this we realized the elements of $\mathfrak{o}^+(2n)_q$ in a quiver (oriented graph) (see [11]). Despite quivers having an associative algebra structure, we are interested only in the embedded linear structure. The associative structure plays a secondary role.

Let Γ_n be a quiver of type A_n. We take two copies of the k-category $k\Gamma_n$: $\mathcal{D}_n = k\Gamma_n \oplus k\Gamma_n$. In the second copy we draw the paths as dashed paths while in the first copy we draw full paths. A dashed path with origin i and terminus j will be labelled S_{ij} and a full path with origin i and terminus j will be labelled M_{ij}. Furthermore, in order to avoid redundancy, instead of drawing two sets of vertices, one for each copy of $k\Gamma_n$, we are drawing just one set of vertices. So the full and dashed paths may have common vertices.

The maps $B^{(k)}$, $k = 1, 2, 3$, appear in tables 1–3, respectively in graphical mode, where a graph of the type of figure 1, for $\lambda \in F$, means the tensorial product $\lambda\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k$.

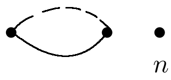
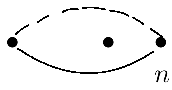
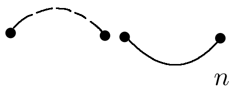
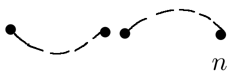
Theorem 1. *The Drinfeld–Jimbo quantum group $U_q^+\mathfrak{o}(8)$ of type D_4 positive is generated by the elements of $\mathfrak{o}^+(8)_q$, with the relations*

$$m(x \otimes y) - m\sigma(x \otimes y) = B^{(1)}(x \otimes y) + mB^{(2)}(x \otimes y) + mm_1B^{(3)}(x \otimes y)$$

where $x, y \in \mathfrak{o}^+(8)_q$ and m is the multiplication map of $U_q^+\mathfrak{o}(8)$.

Proof. Almost all the commutators made of M_{ij}, S_{ij}, $1 \leqslant i < j \leqslant 4$, can be obtained by means of lemmas 1 and 2. The exceptions are M_{13} with S_{23}, M_{13} with S_{12} and S_{12} with S_{23}.

Table 2. The map $B^{(2)}$.

$x \otimes y$	$B^{(2)}(x \otimes y)$
	$(q - q^{-1})$ 
	$-q(q - q^{-1})$ 
	$(q - q^{-1})$ 
	$q^{-1}(q - q^{-1})$ 
	$+(q^{-1} - q)$ 
	$(q - q^{-1})$ 
	$(q - q^{-1})$ 
	$(q^{-1} - q)$ 
Otherwise	0

In fact, in tables 4 and 5 each row represents an isomorphism from $U_q^+(sl_4)$ to a subalgebra of $U_q^+\mathfrak{o}(8)$ sending each head of the table to a generator in $U_q^+\mathfrak{o}(8)$. For instance, from table 4 (second row) and lemma 1 we get

$$M_{13}S_{24} - S_{24}M_{13} = (q - q^{-1})M_{23}S_{14}.$$

There are some canonical basic elements behaving as the generators β_{ij} in lemma 2 but with $1/q$ instead of q. This behaviour is shown in table 6.

In order to deal with the uncovered commutators we define an auxiliary map ρ: ρ coincides with σ, however, when the commutation on $x \otimes y$ has the form $xy - yx = \xi ab$ with $\xi \neq 0$ then the quadratic element $a \otimes b$ is included in ρ. For instance, $\rho(M_{13} \otimes M_{24}) = M_{24} \otimes M_{13} + (q - q^{-1})M_{23} \otimes M_{14}$. Then we can use equation (1) for ρ on $S_{23} \otimes M_{12} \otimes M_{23}$ in order to obtain

$$S_{23}M_{13} - m\rho(S_{23} \otimes M_{13}) = -qS_{13}M_{23} + qm\rho(S_{13} \otimes M_{23})$$

Table 3. The map $B^{(3)}$.

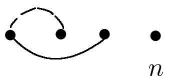
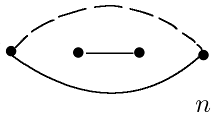
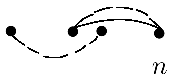
$x \otimes y$	$B^{(3)}(x \otimes y)$
	$q(q-q^{-1})^2$ 
	$-q(q^{-1}-q)^2$ 
Otherwise	0

Table 4. Relations of lemma 1 type.

α_{12}	α_{13}	α_{23}	α_{14}	α_{24}	α_{34}
M_{12}	M_{13}	M_{23}	M_{14}	M_{24}	M_{34}
M_{12}	M_{13}	M_{23}	S_{14}	S_{24}	S_{34}
M_{12}	M_{14}	M_{24}	$-q^{-1}S_{13}$	$-q^{-1}S_{23}$	S_{34}
M_{12}	S_{14}	S_{24}	$-q^{-1}S_{13}$	$-q^{-1}S_{23}$	M_{34}

Table 5. Relations of lemma 2 type.

β_{12}	β_{13}	β_{23}	β_{14}	β_{24}	β_{34}
M_{34}	$-qM_{24}$	M_{23}	S_{23}	S_{24}	S_{34}
S_{34}	$-qS_{14}$	M_{13}	S_{13}	M_{14}	M_{34}
M_{24}	$-qM_{14}$	M_{12}	S_{12}	S_{14}	S_{24}

Table 6. Relations of dual lemma 2 type. The number q is replaced by $1/q$ in lemma 2.

β_{12}	β_{13}	β_{23}	β_{14}	β_{24}	β_{34}
M_{23}	S_{24}	S_{34}	S_{12}	S_{13}	M_{14}
M_{23}	M_{24}	M_{34}	S_{12}	S_{13}	S_{14}

where m is the multiplication map of $U_q^+\mathfrak{o}(8)$. Since $\rho(S_{23}\otimes M_{13}) = qM_{13}\otimes S_{23} + q^2(q-q^{-1})M_{14}\otimes S_{24}$ and $\rho(S_{13}\otimes M_{23}) = q^{-1}M_{23}\otimes S_{13} + (q-q^{-1})M_{24}\otimes S_{14}$ then,

$$S_{23}M_{13} - qM_{13}S_{23} = S_{12} - q^2(q-q^{-1})M_{14}S_{24} - (q-q^{-1})S_{13}M_{23} + q(q-q^{-1})M_{24}S_{14}.$$

In a similar way, using (1) on $S_{23}\otimes M_{24}\otimes S_{14}$ one can prove

$$S_{23}S_{12} - q^{-1}S_{12}S_{23} = (q-q^{-1})^2 S_{13}M_{24}S_{24}$$

and, using (1) on $M_{13}\otimes M_{23}\otimes S_{13}$,

$$M_{13}S_{12} - qS_{12}M_{13} = -q(q-q^{-1})^2 S_{14}M_{23}M_{14}. \qquad \square$$

4. Canonical basic elements in $U_q^+\mathfrak{o}(2n)$

We define inductively the following elements in $U_q^+\mathfrak{o}(2n)$:

$$
\begin{aligned}
&M_{i(i+1)} = E_i \qquad 1 \leqslant i \leqslant n-1\\
&S_{(n-1)n} = E_n\\
&M_{i(i+k)} = M_{i(i+k-1)}M_{(i+k-1)(i+k)} - q^{-1}M_{(i+k-1)(i+k)}M_{i(i+k-1)} \qquad 1 < k < n\\
&S_{(n-1-k)n} = M_{(n-1-k)(n-1-k+1)}S_{(n-1-k+1)n} - q^{-1}S_{(n-1-k+1)n}M_{(n-1-k)(n-1-k+1)}\\
&\qquad\qquad 1 \leqslant k \leqslant n-2\\
&q^{-1}S_{i(i+1)} = M_{in}S_{(i+1)n} - q^{-1}S_{(i+1)n}M_{in} \qquad 1 \leqslant i < n-1\\
&S_{i(i+k)} = M_{i(i+k-1)}S_{(i+k-1)(i+k)} - q^{-1}S_{(i+k-1)(i+k)}M_{i(i+k-1)} \qquad 1 < k < n.
\end{aligned}
$$

Lemma 3.

$$M_{12}M_{1b} - qM_{1b}M_{12} = 0 \qquad 2 < b \leqslant n$$

Proof. This lemma is proved by mathematical induction on b. Let us put

$$[M_{ij}, M_{uv}] = m(Id \otimes Id - \sigma)(M_{ij} \otimes M_{ab})$$

where m is the multiplication map of $U_q^+\mathfrak{o}(2n)$. From the Jacobi identity (1) on $M_{12} \otimes M_{1(b-1)} \otimes M_{(b-1)b}$, we get

$$[M_{12}, [M_{1(b-1)}, M_{(b-1)b}]] - [[M_{12}, M_{1(b-1)}], M_{(b-1)b}] = -q^{-1}[[M_{12}, M_{(b-1)b}], M_{1(b-1)}]. \tag{6}$$

If $b = 3$ then the right-hand side of (6) is $[M_{13}, M_{13}] = 0$ and if $b > 3$ then the right-hand side is zero because $[M_{12}, M_{(b-1)b}] = 0$ by definition. It follows, by mathematical induction on b, that

$$0 = [M_{12}, [M_{1(b-1)}, M_{(b-1)b}]] = [M_{12}, M_{1b}]. \qquad \square$$

Lemma 4.

$$M_{1j}M_{1(j+1)} - qM_{1(j+1)}M_{1j} = 0 \qquad 1 < j \leqslant n \tag{7}$$

$$M_{1j}M_{k(k+1)} - M_{k(k+1)}M_{1k} = 0 \qquad j < k \leqslant n \tag{8}$$

$$M_{1j}S_{(n-1)n} - S_{(n-1)n}M_{1j} = 0. \tag{9}$$

Proof. This lemma is proved by mathematical induction on j. $\square$

Note that (7) is equivalent to

$$M_{1j}M_{j(j+1)}^2 - (q + q^{-1})M_{j(j+1)}M_{1j}M_{j(j+1)} + M_{j(j+1)}^2 M_{1j} = 0$$

therefore, equations (7)–(9) are the defining relations of $U_q^+(\mathfrak{o}(n-j+2))$.

Theorem 2. *Let*

$$\mathcal{B} = \{M_{ij} | 1 \leqslant i < j \leqslant n\} \cup \{S_{ij} | 1 \leqslant i < j \leqslant n\}.$$

If $x, y \in \mathcal{B}$ then there exists a subalgebra A of $U_q^+\mathfrak{o}(2n)$ such that

(i) $x, y \in A$ and $A \simeq U_q^+\mathfrak{o}(8)$ as algebras;

(ii) $m(x \otimes y) - m\sigma(x \otimes y) = B^{(1)}(x \otimes y) + mB^{(2)}(x \otimes y) + mm_1B^{(3)}(x \otimes y)$, where m is the multiplication map of the algebra $U_q^+\mathfrak{o}(2n)$.

Proof.

(i) Assume $x = \gamma_{ab}$ and $y = \rho_{ij}$ with $\gamma = M$ or S and $\rho = M$ or S. If $1 < a, b \leqslant n$ and $1 < i, j \leqslant n$, then x, y are in B, the subalgebra generated by $M_{i(i+1)}$, $1 < i \leqslant n$, and $S_{(n-1)n}$, which is isomorphic to $U_q^+(\mathfrak{o}(n-1))$. Therefore, using mathematical induction, the theorem holds.

We can assume $a = 1$. If $(i, j) \neq (1, 2)$ and $b > 2$, then x, y are in the subalgebra generated by $M_{1b}, M_{bk}, b < k \leqslant n$, and $S_{(n-1)n}$, which is isomorphic to $U_q^+(\mathfrak{o}(n-b+2))$ because of lemma 4. If $(i, j) = (1, 2)$ and $b > 2$ then, since M_{2b}, M_{bn}, S_{bn} are in $B \simeq U_q^+(\mathfrak{o}(n-1))$, we can use mathematical induction and lemma 3 to get that M_{12}, M_{2b}, M_{bn} and S_{bn} are generators of a subalgebra isomorphic to $U_q^+\mathfrak{o}(8)$, which contains x and y.

(ii) If $x, y \in \mathcal{B}$, then, due to (i), there exists A, a subalgebra isomorphic to $U_q^+\mathfrak{o}(8)$, such that $x, y \in A$. Because (ii) holds in A, it also holds in $U_q^+\mathfrak{o}(2n)$. □

5. Generalized Lie algebra structures

Let L be a k-module and $\sigma : L \otimes L \to L \otimes L$ a linear map. We define *the symmetric algebra* $\mathcal{S}(L)$ of L as the factor algebra of the tensor algebra $L^{\otimes}$ by the two-sided ideal generated by

$$x \otimes y - \sigma(x \otimes y) \qquad x, y \in L.$$

Further, suppose that there are $s + 1$ linear maps $B^{(k)} : L \otimes L \to L^{\otimes k} \otimes L^{\otimes}$, $k = 0, 1, \dots, s$. Then define

$$R : L^{\otimes} \otimes L^{\otimes} \to L^{\otimes} \otimes L^{\otimes}$$

by

$$R(u \otimes v) = \begin{cases} v \otimes u & \text{if} \quad u = 1 \quad \text{or} \quad v = 1 \\ \sigma(u \otimes v) + \sum_k B^{(k)}(u \otimes v) \otimes 1 & \text{if} \quad u, v \in L \\ u \otimes v & \text{otherwise.} \end{cases}$$

Let $j : L \to \mathcal{S}(L)$ be the natural embedding and $\hat{L} = L \oplus F$.

Definition 1.

(i) A σ-Lie algebra is an F-module L together with the F-linear maps $\sigma : L \otimes L \to L \otimes L$, $B^{(k)} : L \otimes L \to L^{\otimes k}$, $0 \leqslant k \leqslant s$ and $\_ \cdot \_ : \hat{L} \otimes \mathcal{S}(L) \to \mathcal{S}(L)$ such that

(a) $\sigma^2 = Id$ and $x \cdot 1 = j(x)$ $\forall x \in L$;

(b) $B^{(k)}\sigma = -B^{(k)}$ $k \geqslant 0$;

(c) $(R_1R_2R_1)(x \otimes y \otimes z) \cdot 1 = (R_2R_1R_2)(x \otimes y \otimes z) \cdot 1$, $\forall x, y, z \in L$;

where $(x_1 \otimes \cdots \otimes x_a) \cdot 1 = x_1 \cdot (\cdots (x_a \cdot 1) \cdots)$ and $1 \cdot z = z$, $\forall z \in \mathcal{S}(L)$.

(ii) The universal enveloping algebra $U(L)$ of a σ-Lie algebra L is the factor algebra of $L^{\otimes}$, the tensorial algebra of L, modulo the two-sided ideal generated by

$$x \otimes y - \sigma(x \otimes y) - \sum_{k=0}^{s} B^{(k)}(x \otimes y).$$

Example 1. If we put $B^{(k)} = 0$ for $k \neq 1$ and σ stands for the usual flip $x \otimes y \mapsto y \otimes x$ then we get a classical Lie algebra with bracket $B^{(1)}$ since the so-called braid equation $R_1 R_2 R_1 = R_2 R_1 R_2$ is equivalent, in this case, to the classical Jacobi identity. The map $\_ \cdot \_ : L \otimes \mathcal{S}(L) \to \mathcal{S}(L)$ is the classical representation of the Lie algebra on the symmetric algebra related to the Poincaré–Birkhoff–Witt theorem [20].

It may seem that (i)(c) of definition 1 is an outer equation, however, it is an identity in L because it can be written as

$$\begin{aligned} B^{(1)}B_1^{(1)} - B^{(1)}B_2^{(1)} + B^{(1)}B_2^{(1)}\sigma_1 + B_1^{(0)} - B_1^{(0)}\sigma_2 + B_1^{(0)}\sigma_2\sigma_1 - B_2^{(0)} + B_2^{(0)}\sigma_1 - B_2^{(0)}\sigma_1\sigma_2 \\ = j^{-1}\Bigg(\sigma B_2^{(1)} - \sigma B_1^{(1)} - \sigma B_2^{(1)}\sigma_1 + \sum_{k\neq 1} B^{(k)}\big(B_2^{(1)} - B_2^{(1)}\sigma_1 - B_1^{(1)}\big) \\ + \sum_{k\geqslant 1}\big(B_2^{(k)}\sigma_1\sigma_2 + B_1^{(k)}\sigma_2 - B_1^{(k)}\sigma_2\sigma_1\big) + \sum_{k\geqslant 2}\big(B_2^{(k)} - B_1^{(k)}\big)\Bigg)\cdot 1. \end{aligned} \tag{10}$$

Of course, as Vybornov noted [21], the antisymmetry (i)(b) of definition 1 is equivalent to the invertibility of the map R. Note that our formulation of σ-Lie algebra is related to Vybonov's abstract definition of a quantum Lie algebra.

The map $\_ \cdot \_$ can be constructed in a canonical way for some graded spaces.

Definition 2. *We call a strict grading of a σ-Lie algebra L a direct decomposition of the form*

$$L = \oplus_{i\in\mathbb{N}} L_i$$

where each L_i is a subspace of L such that

$$B^{(k)}(L_i \otimes L_j) \subseteq \oplus_{u_1+\cdots+u_k\leqslant i+j-1} L_{u_1} \otimes \cdots \otimes L_{u_k} \qquad \forall i, j \in \mathbb{N}. \tag{11}$$

As usual, if $x \in L_i$, x is said to be homogeneous of degree i and we put $\eta(x) = i$.

Suppose that for each $i \in \mathbb{N}$ there is a basis $\mathcal{B}_i$ consisting of homogeneous elements of degree i, and moreover that $\mathcal{B} = \cup_{i\in\mathbb{N}}\mathcal{B}_i$ is a totally ordered set.

If $\Sigma = (x_1, \ldots, x_k)$ is a finite non-decreasing sequence of basic elements of $\mathcal{B}$, we put $z_\Sigma = j(x_1)\cdots j(x_k) \in \mathcal{S}(L)$, $\eta(\Sigma) = \eta(x_1) + \cdots + \eta(x_n)$ and $z_\emptyset = 1 \in \mathcal{S}(L)$, $\eta(z_\emptyset) = 0$. Furthermore, $x \leqslant \Sigma = (x_1, \ldots, x_k)$, for $x \in \mathcal{B}$, means $x \leqslant x_1$.

Lemma 5. *Let $\mathcal{S}_p$ be the submodule of $\mathcal{S}(L)$ generated by z_Σ such that $\eta(\Sigma) \leqslant p$.*

If $\sigma(x \otimes y) = q_{xy} y \otimes x, \forall x, y \in \mathcal{B}$, then there exists a k-morphism $\_ \cdot \_ : L \otimes \mathcal{S}(L) \to \mathcal{S}(L)$ such that, for any $x_\lambda \in \mathcal{B}$,

(i) $x_\lambda \cdot z_\Sigma = j(x_\lambda) z_\Sigma$ if $x_\lambda \leqslant \Sigma$;
(ii) $x_\lambda \cdot z_\Sigma - j(x_\lambda) z_\Sigma \in \mathcal{S}_{\eta(x_\lambda)+\eta(\Sigma)-1}$.

Proof. A subset of $\{z_\Sigma | \Sigma$ is a non-decreasing sequence of basic elements of $L\}$ is a basis of $\mathcal{S}(L)$. So, we are going to define $\_ \cdot \_$ on such a subset. We proceed with mathematical induction on $\eta(x_\lambda) + \eta(\Sigma)$. If $\eta(x_\lambda) + \eta(\Sigma) = 1$ then $\Sigma = \emptyset$. Then define $x_\lambda \cdot z_\emptyset = j(x_\lambda)$. Now assume that $x_{\lambda'} \cdot z_{\Sigma'}$ is defined for $\eta(x_{\lambda'}) + \eta(\Sigma') < \eta(x_\lambda) + \eta(\Sigma)$, satisfying (i) and (ii). We have to define $x_\lambda \cdot z_\Sigma$. There are two cases: $x_\lambda \leqslant \Sigma$ or $x_\lambda \nleqslant \Sigma$.

Case $\lambda \leqslant \Sigma$: define $x_\lambda \cdot \Sigma = j(x_\lambda) z_\Sigma$.

Case $\lambda \nleqslant \Sigma$: we may write $\Sigma = (x_\mu, N)$ where $x_\mu \leqslant N$ and $x_\lambda > x_\mu$ and

$$B^{(k)}(x_\lambda \otimes x_\mu) = \sum_i \xi_i x_{i_1} \otimes \cdots \otimes x_{i_k}$$

where each $x_{i_j} \in \mathcal{B}$ and $\xi_i \in F$. Because of the induction hypothesis and (11) we can put $w = x_\lambda \cdot z_N - z_\lambda z_N \in \mathcal{S}_{\eta(x_\lambda)+\eta(N)-1}$. Then $x_\mu \cdot w$ is already defined and we can also define

$$B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N = \sum_i \xi_i x_{i_1} \cdot \big(\cdots \big(x_{i_k} \cdot z_N\big) \cdots\big).$$

Therefore we can define

$$x_\lambda \cdot z_\Sigma = j(x_\lambda) j(x_\mu) z_N + q_{\lambda\mu} x_\mu \cdot w + \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N \tag{12}$$

satisfying (ii). □

Example 2. Let V be an F-vectorial space and $f : V \otimes V \to k$ a symmetric bilinear form. Now we define $B^{(0)} = 2f, \sigma : V \otimes V \to V \otimes V, x \otimes y \mapsto -y \otimes x$, and a trivial decomposition $V = \oplus_i V_i, V_1 = V, V_i = 0$ if $i > 0$. We get $R_1 R_2 R_1 = R_2 R_1 R_2$. Then, the universal enveloping algebra of the σ-Lie algebra $(V, \sigma, B^{(0)})$ is the classical Clifford algebra $Cl(V)$ of V. Just as in the case of Lie algebras, the linear map $\_ \cdot \_$ is the classical representation on the symmetric algebra related to the Poincaré–Birkhoff–Witt theorem for Clifford algebras.

Example 3. The space $\mathfrak{o}^+(8)_q$ has a structure of σ-Lie algebra; the equation of (i)(c) of the definition is obtained by straightforward calculations (using *Mathematica* [22]). The decomposition (11) is induced by defining the canonical basic elements of $\mathfrak{o}^+(8)_q$ as homogeneous with the following degrees: $\eta(M_{12}) = 3, \eta(M_{13}) = 3, \eta(M_{14}) = 1, \eta(S_{14}) = 1, \eta(S_{12}) = 1, \eta(S_{13}) = 1, \eta(M_{23}) = 1, \eta(M_{24}) = 1, \eta(S_{24}) = 1, \eta(S_{23}) = 3, \eta(M_{34}) = 3, \eta(S_{34}) = 3$. From theorem 1 we get $U(\mathfrak{o}^+(8)_q) \simeq U_q^+\mathfrak{o}(8)$ as associative algebras.

Proposition 2. *Let L be a σ-Lie algebra with strict grading $L = \oplus_{i\in\mathbb{N}} L_i$, let $\mathcal{B}_i$ be a basis of L_i, $\forall i \in \mathbb{N}$, and $\mathcal{B} = \cup_{i\in\mathbb{N}} \mathcal{B}_i$. There exists a linear map $\_ \cdot \_ : L \otimes \mathcal{S}(L) \to \mathcal{S}(L)$ such that*

$$x_\lambda \cdot (x_\mu \cdot z_N) - \sigma(x_\lambda \otimes x_\mu) \cdot z_N = \sum_{k=0}^{s} B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N \qquad \forall x_\lambda, x_\mu \in \mathcal{B}. \tag{13}$$

Proof. Natural modifications to the proof of lemma 1 in [20] can be done for the σ-Lie algebra case (modifications of this kind were done in lemma V.2 of [23] for some generalized Lie algebras called T-Lie algebras). Let $\_ \cdot \_$ be the map defined in lemma 5. There are two cases:

(i) $x_\mu \leqslant N$ or $x_\lambda \leqslant N$;
(ii) $x_\mu \nleqslant N$ and $x_\lambda \nleqslant N$.

(i) Since antisymmetry holds we can assume $x_\mu \leqslant N$ and $x_\mu < x_\lambda$. Let $M = (x_\mu, N)$; then by definition (12),

$$\begin{aligned} &q_{\lambda\mu} x_\mu \cdot (x_\lambda \cdot z_N) + \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N \\ &\qquad = q_{\lambda\mu} x_\mu \cdot (j(x_\lambda) z_N) + q_{\lambda\mu} x_\mu \cdot (x_\lambda \cdot z_N - j(x_\lambda) z_N) + \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N \\ &\qquad = x_\lambda \cdot z_M = x_\lambda \cdot (x_\mu \cdot z_N). \end{aligned}$$

(ii) Let $N = (x_\gamma, Q)$ where $x_\gamma \leqslant Q, x_\gamma < x_\lambda$ and $x_\gamma < x_\mu$. Suppose that (13) holds for any $\eta(x'_\lambda) + \eta(x'_\mu) + \eta(N') \leqslant r$. For $\eta(x_\lambda) + \eta(x_\mu) + \eta(N) \leqslant r + 1$ we have $x_\mu \cdot z_Q = j(x_\mu) z_Q + w$, where $w \in \mathcal{S}_{\eta(x_\mu)+\eta(Q)-1}$. Then,

$$
\begin{aligned}
x_\lambda \cdot (x_\mu \cdot z_N) &= x_\lambda \cdot q_{\mu\gamma}(x_\gamma \otimes x_\mu) \cdot z_Q + x_\lambda \cdot \sum_k B^{(k)}(x_\mu \otimes x_\gamma) \cdot z_Q \\
&= q_{\mu\gamma} q_{\lambda\gamma} x_\gamma \cdot (x_\lambda \cdot (x_\mu \cdot z_Q)) + q_{\mu\gamma} \sum_k B^{(k)}(x_\lambda \otimes x_\gamma) \cdot (x_\mu \cdot z_Q) \\
&\quad + x_\lambda \cdot \sum_k B^{(k)}(x_\mu \otimes x_\gamma) \cdot z_Q
\end{aligned}
$$

since, by calculation of the degree, (13) can be applied to $x_\lambda \cdot (x_\gamma \cdot j(x_\mu) z_Q) + x_\lambda \cdot (x_\gamma \cdot w) = x_\lambda \cdot (x_\gamma \cdot (x_\mu \cdot z_Q))$. Using that x_λ, x_μ are interchangeable and $\eta(x_\lambda) + \eta(x_\mu) + \eta(z_Q) \leqslant r$ it follows:

$$
\begin{aligned}
x_\lambda \cdot (x_\mu \cdot z_N) - q_{\lambda\mu} x_\mu \cdot (x_\lambda \cdot z_N) &= \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N + \sum_k \big(\big(B_2^{(k)} \sigma_1 \sigma_2 - B_1^{(k)}\big) \\
&\quad + B_1^{(k)} \sigma_2 - B_2^{(k)} \sigma_1 + \big(B_2^{(k)} - B_1^{(k)} \sigma_2 \sigma_1\big)\big)(x_\lambda \otimes x_\mu \otimes x_\gamma) \cdot z_Q \\
&= \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N + ((R_2 R_1 R_2 - R_1 R_2 R_1)(x_\lambda \otimes x_\mu \otimes x_\gamma)) \cdot z_Q.
\end{aligned}
$$

Now, we can put equation (10) on $x_\lambda \otimes x_\mu \otimes x_\gamma$ as $A \cdot 1 = B \cdot 1$ where $A \in L$ and $B \in L^{\otimes}$; then using the induction hypothesis, (10) means that $A \equiv B \pmod{J_s}$, where J_s is the two-sided ideal of the tensor algebra $L^{\otimes}$ generated by $x \otimes y - \sigma(x \otimes y) - \sum_k B^{(k)}(x \otimes y)$, such that $\eta(x) + \eta(y) < s = \eta(x_\lambda) + \eta(x_\mu) + \eta(x_\gamma)$.

Straightforward calculations show that

$$
((R_2 R_1 R_2 - R_1 R_2 R_1)(x_\lambda \otimes x_\mu \otimes x_\gamma)) \cdot z_Q = A \cdot z_Q - B \cdot z_Q = A \cdot z_Q - A \cdot z_Q
$$

since $B \equiv A \pmod{J_s}$ and using the induction hypothesis again.

Therefore

$$
x_\lambda \cdot (x_\mu \cdot z_N) - q_{\lambda\mu} x_\mu \cdot (x_\lambda \cdot z_N) = \sum_k B^{(k)}(x_\lambda \otimes x_\mu) \cdot z_N.
$$

□

Paralleling the classical Lie algebra theory, one can prove

Theorem 3. *Let L be a σ-Lie algebra with strict grading $L = \oplus_{i \in \mathbb{N}} L_i$. Then*

$$
U(L) \simeq \mathcal{S}(L)
$$

as F-modules.

6. Morphisms and integer programming problems

Definition 3. *If $\big(L_i, \sigma_i, B_i^{(k)}\big)$ is a σ-Lie algebra, $i = 1, 2$, then a linear morphism $\varphi : L_1 \to L_2$ is called a σ-Lie algebra morphism if the following diagrams commute:*

$$
\begin{array}{ccc}
L_1 \otimes L_1 & \xrightarrow{B_1^{(k)}} & L_1^{\otimes k} \\
{\scriptstyle \varphi\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \varphi^{\otimes k}} \\
L_2 \otimes L_2 & \xrightarrow{B_1^{(k)}} & L_2^{\otimes k}
\end{array}
\qquad k = 0, 1, \ldots \qquad
\begin{array}{ccc}
L_1 \otimes L_1 & \xrightarrow{\sigma_1} & L_1 \otimes L_1 \\
{\scriptstyle \varphi\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \varphi\otimes\varphi} \\
L_2 \otimes L_2 & \xrightarrow{\sigma_2} & L_2 \otimes L_2
\end{array}
$$

where $\varphi^{\otimes k} : L_1^{\otimes} \to L_2^{\otimes}$, $\varphi^{\otimes k} = \varphi \otimes \cdots \otimes \varphi$ (k factors).

Fix $1 \leqslant k < n - 1$. Let $\mathfrak{o}^+(2n, k)_q$ be the subalgebra of $\mathfrak{o}^+(2n)_q$ obtained by dropping the basic elements M_{ik}, M_{kj} and S_{ik}, S_{kj} for $1 \leqslant i < k < j \leqslant n$.

Proposition 3. *There exists an isomorphism $\varphi : \mathfrak{o}^+(2n, k)_q \to \mathfrak{o}^+(2(n-1))_q$ of σ-Lie algebras such that*

(i) $\varphi(M_{(k-1)(k+1)}) = M_{(k-1)k}$ *and* $\varphi(S_{(k-1)(k+1)}) = S_{(k-1)k}$;
(ii) φ *is order preserving.*

Proof. Let $f : \{1, \ldots, \hat{k}, \ldots, n\} \to \{1, \ldots, n-1\}$, defined by

$$f(j) = \begin{cases} j & \text{if} \quad j < k \\ j-1 & \text{if} \quad j > k \end{cases}$$

then we define $\varphi : \mathfrak{o}^+(2n, k)_q \to \mathfrak{o}^+(2(n-1))_q$, $\varphi(M_{ij}) = M_{f(i)f(j)}$, $\varphi(S_{ij}) = S_{f(i)f(j)}$.

Suppose that $k < i$ in (4). Then,

$$\varphi(M_{i(i+1)}) = M_{(i-1)i} > \cdots > \varphi(M_{in}) = M_{(i-1)(n-1)} > \varphi(S_{in}) = S_{(i-1)(n-1)} > \varphi(S_{i(i+1)}) \\ = S_{(i-1)i} > \cdots > \varphi(S_{i(n-1)}) = S_{(i-1)(n-2)}$$

which is the order definition (4) in $\mathfrak{o}^+(2(n-1))_q$. In a similar way, if $k > i$,

$$\cdots > M_{i(k-1)} > M_{ik} > M_{i(k+1)} > \cdots M_{in} > S_{in} > \cdots > S_{i(k-1)} > S_{ik} > S_{i(k+1)} > \cdots$$

then,

$$\cdots > \varphi(M_{i(k-1)}) = M_{i(k-1)} > \varphi(M_{i(k+1)}) = M_{ik} > \cdots > \varphi(M_{in}) = M_{i(n-1)} > \varphi(S_{in}) \\ = S_{i(n-1)} > \cdots > \varphi(S_{i(k-1)}) = S_{i(k-1)} > \varphi(S_{i(k+1)}) = S_{ik} > \cdots.$$

Again, this is the order (4) in $\mathfrak{o}^+(2(n-1))_q$.

Now, suppose $k < j$ in (5). Then,

$$\varphi(S_{j(n-1)}) = S_{(j-1)(n-2)} > \varphi(M_{(j+1)(j+2)}) = M_{j(j+1)}$$

and, if $k > j$,

$$\varphi(S_{j(n-1)}) = S_{j(n-2)} > \varphi(M_{(j+1)(j+2)}) = M_{(j+1)(j+2)}$$

(since $j+1 \neq k \neq j+2$) which are the order definition (5) in $\mathfrak{o}^+(2(n-1))_q$.

Therefore φ is order preserving.

Let us take β_{ij}, γ_{ab} canonical basic elements ($\beta = M$ or S and $\gamma = M$ or S). From tables 1 and 2 we learn that the coefficient of $B^{(k)}(\beta_{ij} \otimes \gamma_{ab})$ does not depend on ij, ab ($k = 1, 2, 3$). Further, if we drop a vertex distinct from i, j, a and b, the paths do not change the graph shape. Therefore

$$B^{(1)}(\varphi \otimes \varphi)(\beta_{ij} \otimes \gamma_{ab}) = \varphi B^{(1)}(\beta_{ij} \otimes \gamma_{ab})$$
$$B^{(2)}(\varphi \otimes \varphi)(\beta_{ij} \otimes \gamma_{ab}) = (\varphi \otimes \varphi) B^{(2)}(\beta_{ij} \otimes \gamma_{ab})$$
$$B^{(3)}(\varphi \otimes \varphi)(\beta_{ij} \otimes \gamma_{ab}) = (\varphi \otimes \varphi \otimes \varphi) B^{(3)}(\beta_{ij} \otimes \gamma_{ab}).$$

□

In order to define a σ-Lie algebra structure over the elements E_{ij}, S_{ij}, $1 \leqslant i < j \leqslant n$, first we have to define a strict grading. It suffices to define some degrees on them, satisfying the linear inequalities given by tables 1–3: in each row of any table, the sum of the degrees of the elements in the first column has to be greater than the sum of the degrees of the second column. For instance, from 3, first row,

$$\eta(M_{ij}) + \eta(S_{ib}) > \eta(M_{in}) + \eta(M_{jb}) + \eta(S_{in}) \qquad \text{if} \quad j < b < n.$$

Finding a solution to such a system is known as an *integer programming problem* (with the aim function given by the trivial zero function); if a solution does exist then the inequality system is called feasible. Note that if $\eta(M_{ij})$, $\eta(S_{ij})$, $1 \leqslant i < j \leqslant n$, is a solution for the case n, then constraint to $2 \leqslant i < j \leqslant n$ gives a solution for the case $n-1$. For example, since solutions for $n = 6$ can be found using computer packages, the map $\_ \cdot \_$ from lemma 5 can be defined also for $n = 4$ and 5 and the Jacobi identity (i)(c) can be proved by straightforward calculations.

Proposition 4. *If $n = 4, 5$ and 6 then $\mathfrak{o}^+(2n)_q$ has a structure of σ-Lie algebra.*

Although the following lemma is trivial, it is useful in order to prove the generalized Jacobi identity on $\mathfrak{o}^+(2n)_q$, when the system is feasible.

Lemma 6. *If $\gamma_{ij}, \gamma_{ab}, \gamma_{uv}$ are paths of $\mathcal{D}_n$, then there exists a subalgebra B of $\mathcal{D}_n$ such that*

(i) $\gamma_{ij}, \gamma_{ab}, \gamma_{uv} \in B$;

(ii) $B \overset{\phi}{\simeq} \mathcal{D}_6$;

(iii) $\phi(\gamma_{ij}), \phi(\gamma_{ab}), \phi(\gamma_{uv})$ *are paths in* $\mathcal{D}_6$.

Proposition 5. *Let $n \geqslant 4$. If the inequality system described above is feasible, then $\mathfrak{o}^+(2n)_q$ has a strict grading and there exists a linear map $\_ \cdot \_ : L \otimes \mathcal{S}(\mathfrak{o}^+(2n)_q) \to \mathcal{S}(\mathfrak{o}^+(2n)_q)$, such that the Jacobi identity (i)(c) of definition 1 holds.*

Proof. The solution to the integer programming problem ensures the existence of the map $\_ \cdot \_$. For $n = 4, 5$ and 6 the generalized Jacobi identity (i)(c) of definition 1 can be proved by straightforward calculations. For $n > 6$ we can take x, y, z canonical basic elements of $\mathfrak{o}^+(2n)_q$. Then, from lemma 6 and proposition 3, there exists a space L_0 which is a σ-Lie algebra such that

(i) $x, y, z \in L_0$;

(ii) $L_0 \overset{\varphi}{\simeq} \mathfrak{o}^+(12)_q$ as σ-Lie algebras.

From the generalized Jacobi identity on $\mathfrak{o}^+(12)_q$, it follows the Jacobi identity on x, y, z. □

If we put $q = 1$ then $B^{(2)} = 0$ and $B^{(3)} = 0$ while $B^{(1)}$ is the classical bracket of $\mathfrak{o}^+(2n)$ where $M_{ij}|_{q=1} = e_{(i+n)(j+n)} - e_{ji}$ and $S_{ij}|_{q=1} = e_{(i+n)j} - e_{(j+n)i}$ (e_{uv}, $1 \leqslant u, v \leqslant 2n$, the canonical basis of gl_{2n}).

Theorem 4.

(i) $U(\mathfrak{o}^+(2n)_q) \simeq U_q^+\mathfrak{o}(2n)$ *as associative algebras;*

(ii) $\dim_F \mathfrak{o}^+(2n)_q = n^2 - n$;

(iii) if $q \to 1$ *then* $\mathfrak{o}^+(2n)_q \to \mathfrak{o}^+(2n)$.

(iv) If the inequality system described above is feasible, then

- *(a) a structure of σ-Lie algebra can be defined on $\mathfrak{o}^+(2n)_q$;*
- *(b) the universal enveloping algebra $U_q^+\mathfrak{o}(2n)$ has a linear basis formed by the monomials made of finite non-decreasing sequences of canonical basic elements of $\mathfrak{o}^+(2n)_q$.*

7. About other quantum Lie algebras

The quantum Lie algebras of [16] are finite-dimensional spaces with a binary operation (called quantum Lie bracket) which are invariant under the adjoint representation of the Hopf algebra structure (ad-invariance) and also have a q-antisymmetry property. Of course, q-antisymmetry is a generalization of the classical antisymmetry property, while one could say that the generalized Jacobi identity is the property of the quantum Lie bracket of being a module morphism on the quantum group. Actually, such a property is used to obtain a grading of the quantum Lie algebras by *quantum* roots, just as the classical Jacobi identity is used to obtain a grading of the Lie algebras by its roots. However, a polynomial style quantum Jacobi identity is expected.

Our Jacobi identity (definition 1 (i)(c)) is of this type. Therefore, a natural question is if Delius *et al*'s quantum Lie algebra satisfies it. The answer is negative, at least in a direct way. Because our definition of the Jacobi identity uses a generalization of the classical flip $\rho : x \otimes y \mapsto y \otimes x$ and, besides a binary operation, additional higher degree operators: $B^{(k)} : L^{2\otimes} \to L^{k\otimes}, k = 2, 3$. The flip used in [16] is the classical one, since the quantum groups used there are Hopf algebras which are not braided. For instance, the adjoint representation is defined by

$$ad(x)(y) = \rho_2 S_2(\Delta \otimes Id)(x \otimes y) \tag{14}$$

where Δ is the coproduct and S is the antipode. Besides there are no q-commutators of higher order; this is a remarkable fact in the theory of [16], but from our point of view this leads us to a classical Jacobi identity which does not hold in the quantum Lie algebras of [16].

Reciprocally, our quantum Lie algebras do not satisfy ad-invariance. For example, the vectorial subspace $\left(sl^{+}_{n+1}\right)_q$ generated by M_{ij} $1 \leqslant i < j \leqslant n+1$, is a σ-Lie algebra of type A_n positive. This means that $U\left(\left(sl^{+}_{n+1}\right)_q\right)$ is an associative algebra isomorphic to $U^{+}_q(sl_{n+1})$ which is the positive part of the Drinfeld–Jimbo quantum group of type A_n positive. In fact $U^{+}_q(sl_{n+1})$ is a *braided* Hopf algebra [24], with a non-involutive braid given by

$$\sigma(M_{ij} \otimes M_{ab}) = q^{c_{ij,ab}} M_{ab} \otimes M_{ij}$$

where $c_{ij,ab} = \delta_{ia} - \delta_{ib} - \delta_{ja} + \delta_{jb}$ (see [18]). We can calculate the braided version of (14) on $M_{13} \otimes M_{24}$, with σ instead of ρ and Δ the braided coproduct of $U^{+}_q(sl_{n+1})$, in order to obtain that

$$\phi(M_{ij}) = M_{ij} \otimes 1 + 1 \otimes M_{ij} + (Id - \sigma^2) \sum_{i<k<j} M_{ik} \otimes M_{kj}$$

then $ad(M_{13})(M_{24}) \notin \left(sl^{+}_{n+1}\right)_q$.

Also, there is not a direct link to the quantum Lie algebras of [10]. It can be proved that $U^{+}_q\mathfrak{o}(2n)$ is a braided Hopf algebra with braid given by the non-involutive operator τ,

$$\tau(E_i \otimes E_j) = q^{a_{ij}} E_j \otimes E_i$$

coproduct ϕ such that each generator E_i is a primitive element, where $C = (a_{ij})$ is the Cartan matrix of type D_n. The map τ can be extended to $U^{+}_q\mathfrak{o}(2n)$ by means of the following equations:

$$\tau\rho = \tau \qquad \tau m_1 = m_2\tau_1\tau_2 \qquad \tau m_2 = m_1\tau_2\tau_1$$

where m is the multiplication map of $U^{+}_q\mathfrak{o}(2n)$, because C is a symmetric matrix. Note that according to the deformation theory [25], in order to get a deformation of a Hopf algebra it suffices to deform the product and the coproduct.

Therefore $\phi\left(\left(sl^{+}_{n+1}\right)_q\right) \subset \left(sl^{+}_{n+1}\right)_q \otimes \left(sl^{+}_{n+1}\right)_q$. However, $\phi(\mathfrak{o}^{+}(2n)_q) \not\subset \mathfrak{o}^{+}(2n)_q \otimes \mathfrak{o}^{+}(2n)_q$ because, for instance, $\phi(S_{23}) \notin \mathfrak{o}^{+}(2n)_q \otimes \mathfrak{o}^{+}(2n)_q$. We conclude that $\mathfrak{o}^{+}(2n)_q$ is not a braided Lie algebra.

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